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Various kinds of semicontinuity and the solution sets of parametric multivalued symmetric vector quasiequilibrium problems

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Abstract We introduce some definitions related to semicontinuity of multivalued mappings and discuss various kinds of semicontinuity-related properties. Sufficient conditions for the solution sets of parametric multivalued symmetric vector quasiequilibrium problems to have these properties are established. Comparisons of the solution sets of our two problems are also provided. As an example of applications of our main results, the mentioned semicontinuity-related properties of the solution sets to a lower and upper bounded quasiequilibrium problem are obtained as consequences.

Keywords U-lower (or upper)-level closednesss · U-Hausdorff-lower (or upper)- level closedness · U-lower (or upper)-semicontinuity · U-Hausdorff-lower (or upper)-semicontinuity · (Hausdorff) lower or upper semicontinuity · U-inclusion property · Symmetric quasiequilibrium problems · Lower and upper bounded quasiequilibrium problems · Solution sets

1 Introduction

The equilibrium problem, introduced in Blum and Oettli (1994), has been being studied intensively so far with more and more general problem settings to include various practical optimization-related problems. The first main focus has been made for existence conditions, see e.g., recent papers and references therein: Bianchi and Schaible (2004), Iusem and Sosa (2003), and Hai and Khanh (2007a) for equilibrium problems, Tan (2004), Luc and Tan (2004), and Hai and Khanh (2007b) for variational inclusion problems, Ansari et al. (2000,

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2002), Lin (2006), and Hai and Khanh (2006) for systems of equilibrium problems and Hai and Khanh (2007c) for systems of variational inclusion problems. Recently, to model generally symmetric features in varying problems in practice, a symmetric quasiequilibrium problem was proposed in Noor and Oettli (1994). This result was extended to the vector case in Fu (2003), and Farajzadeh (2006) and to the multivalued case in Anh and Khanh (2007c).

Stability is a vital subject of applied mathematics. However, for the above-mentioned problems there have been limited number of works in the literature, see Bianchi and Pini (2003, 2006), Anh and Khanh (2004, 2006, 2007a), submitted for publication Ait Mansour and Riahi (2005), and Haung et al. (2006). To the best of our knowledge, no paper has been devoted to stability of symmetric equilibrium problems. This motivates our commitment in this note: investigating semicontinuity of the solution sets of these problems at a general setting. Moreover, we try to highlight kinds of semicontinuity, proposing also some semicontinuity-related definitions to have a better insight. We pay attention on relationships of kinds of semicontinuity-related properties too.

In the sequel, if not otherwise stated, let X, Y, and Z be Hausdorff topological vector spaces. Let Λ and M be topological spaces. Let $K \subseteq X$, $D \subseteq Y$ be nonempty. Let $C \subseteq Z$ be closed with nonempty interior int$C$. Let $S, A : K \times D \times \Lambda \to 2^K$, $T, B : K \times D \times \Lambda \to 2^D$, $F : K \times D \times K \times M \to 2^Z$ and $G : D \times K \times D \times M \to 2^Z$ be multivalued mappings. The parametric symmetric quasiequilibrium problems under our consideration consist of, for $(\lambda, \mu) \in \Lambda \times M$,

(SQEP$_1$) finding $(\bar{x}, \bar{y}) \in K \times D$ such that $\bar{x} \in S(\bar{x}, \bar{y}, \lambda)$, $\bar{y} \in T(\bar{x}, \bar{y}, \lambda)$, and

$$F(x, \bar{y}, x^*, \mu) \cap (Z \backslash -\text{int}C) \neq \emptyset, \forall x \in S(\bar{x}, \bar{y}, \lambda), \forall x^* \in A(\bar{x}, \bar{y}, \lambda),$$
$$G(y, \bar{x}, y^*, \mu) \cap (Z \backslash -\text{int}C) \neq \emptyset, \forall y \in T(\bar{x}, \bar{y}, \lambda), \forall y^* \in B(\bar{x}, \bar{y}, \lambda);$$

(SQEP$_2$) finding $(\bar{x}, \bar{y}) \in K \times D$ such that $\bar{x} \in S(\bar{x}, \bar{y}, \lambda)$, $\bar{y} \in T(\bar{x}, \bar{y}, \lambda)$, and

$$F(x, \bar{y}, x^*, \mu) \subseteq Z \backslash -\text{int}C, \forall x \in S(\bar{x}, \bar{y}, \lambda), \forall x^* \in A(\bar{x}, \bar{y}, \lambda),$$
$$G(y, \bar{x}, y^*, \mu) \subseteq Z \backslash -\text{int}C, \forall y \in T(\bar{x}, \bar{y}, \lambda), \forall y^* \in B(\bar{x}, \bar{y}, \lambda).$$

Note that sufficient conditions for the solution existence of these problems were provided in Anh and Khanh (2007c). Therefore, we now focus only on the solution stability, assuming that the referred solution always exists. Notice also that our problem setting includes all that of Noor and Oettli (1994), Fu (2003), and Farajzadeh (2006) for symmetric quasiequilibrium problems and hence of course that of quasiequilibrium problems (when $Y = X$, $G(y, \bar{x}, y^*) \equiv C$, $B(x, y) = D$ and $T(x, y) = clS(x, y)$).

The layout of the paper is as follows. We supply some definitions and preliminaries in the rest of this section. In Sect. 2, we derive various kinds of semicontinuity of multivalued mappings and the relations of this concepts. Section 3 is devoted to kinds of lower semicontinuity of the solution sets, while different types of upper semicontinuity are the subjects of Sect. 4. In the next Sect. 5 we discuss some comparisons of the solution sets of our two problems. Applications to a lower and upper bounded quasiequilibrium problem are presented in the final Sect. 6.

Recall now some notions. Let X and Y be as above and $Q : X \to 2^Y$ be a multifunction. Q is called lower semicontinuous (lsc) at x_0 if: $Q(x_0) \cap U \neq \emptyset$ for some open subset $U \subseteq Y$ implies the existence of a neighborhood N of x_0 such that, $\forall x \in N$, $Q(x) \cap U \neq \emptyset$. Q is upper semicontinuous (usc) at x_0 if for each open subset $U \supseteq Q(x_0)$, there is a neighborhood N of x_0 such that $U \supseteq Q(N)$. Q is said to be Hausdorff lower semicontinuous (H-lsc) at x_0 if for each neighborhood B of the origin in Y, there is a neighborhood N of x_0 such that $Q(x_0) \subseteq Q(x) + B, \forall x \in N$. Q is termed Hausdorff upper semicontinuous (H-usc) at x_0 if the last inclusion is replaced by $Q(x) \subseteq Q(x_0) + B, \forall x \in N$. Q is called closed at x_0 if, for each net $(x_\alpha, y_\alpha) \in \text{graph}Q := \{(x, y) \mid y \in Q(x)\} : (x_\alpha, y_\alpha) \to (x_0, y_0), y_0 \in Q(x_0)$. We say that Q satisfies a certain property in a subset $A \subseteq X$ if Q satisfies it at every point of A. If $A = \text{dom}Q := \{x \mid Q(x) \neq \emptyset\}$ we omit "in dom$Q$" in the saying.

The following assertions are known and we give a reference only in cases of nonpopular statements.

(a) Q is lsc at x_0 if and only if $\forall x_\alpha \to x_0$. $\forall y \in Q(x_0), \exists y_\alpha \in Q(x_\alpha), y_\alpha \to y$.
(b) Q is closed if and only if graph$Q$ is closed.
(c) Q is closed at x_0 if Q is H-usc at x_0 and $Q(x_0)$ is closed (Anh and Khanh 2004).
(d) Q is H-usc at x_0 if Q is usc at x_0. Conversely, Q is usc at x_0 if Q is H-usc at x_0 and $Q(x_0)$ is compact (Anh and Khanh 2004).
(e) Q is usc at x_0 if $Q(A)$ is compact for any compact subset A of dom$Q$ and Q is closed at x_0.
(f) Q is usc at x_0 if Y is compact and Q is closed at x_0.
(g) Q is lsc at x_0 if Q is H-lsc at x_0. The converse is true if $Q(x_0)$ is compact (Hu and Papageorgiou 1997).

2 Various kinds of semicontinuity

We propose some definitions related to semicontinuity to have a better insight as follows.

Definition 2.1 Let X be a Hausdorff topological space, Y be a topological vector space, $Q : X \to 2^Y$ and $\emptyset \neq U \subseteq Y$.

(i) Q is called U-lower-level closed at x_0 if $Q(x_0) \subseteq \text{cl}U$ whenever $Q(x_\alpha) \subseteq \text{cl}U, \forall\alpha$ for some net $x_\alpha \to x_0$ (cl(.) means the closure of (.)).
(ii) Q is said to be U-Hausdorff-lower-level closed at x_0 if there is $\bar{\alpha}$, $Q(x_0) \setminus \text{cl}U \subseteq Q(x_{\bar{\alpha}}) + B$ whenever a net $x_\alpha \to x_0$ and B is a neighborhood of 0.
(iii) Q is said to be U-upper-level closed at x_0 if $Q(x_0) \nsubseteq -\text{int}U$ whenever $Q(x_\alpha) \nsubseteq -\text{int}U, \forall\alpha$, for some net $x_\alpha \to x_0$.
(iv) Q is termed U-Hausdorff-upper-level closed at x_0 if, for each neighborhood B of 0, $Q(x_0) + B \nsubseteq -\text{int}U$ whenever a net $x_\alpha \to x_0$ exists with $Q(x_\alpha) \nsubseteq -\text{int}U, \forall\alpha$.
Note that if int$U = \emptyset$ then each Q satisfies both (iii) and (iv). Furthermore, recall that Q is U-lower-level closed means that Q is U-lower-level closed at every $x \in \text{dom}Q$.

Next we define other relaxed semicontinuity properties.

Definition 2.2 Let X, Y, Q and U be as in Definition 2.1.

(i) Q is said to be U-lower semicontinuous (U-lsc) at x_0 if

$$[x_\alpha \to x_0, Q(x_0) \cap \text{int}U \neq \emptyset] \Longrightarrow [\exists\bar{\alpha}, Q(x_{\bar{\alpha}}) \cap \text{int}U \neq \emptyset].$$

(ii) Q is said to be U-Hausdorff-lower semicontinuous (U-Hlsc) at x_0 if, for any $x_\alpha \to x_0$ and B (a neighborhood of 0 in Y), there is $\bar{\alpha}$ such that $Q(x_0) \cap \text{int}U \subseteq Q(x_{\bar{\alpha}}) + B$.
(iii) Q is called U-upper semicontinuous (U-usc) at x_0 if

$$[x_\alpha \to x_0, Q(x_0) \subseteq \text{int}U] \Longrightarrow [\exists\bar{\alpha}, Q(x_{\bar{\alpha}}) \subseteq \text{int}U].$$

(iv) Q is termed U-Hausdorff-upper semicontinuous (U-Husc) at x_0 if, for

$$[x_\alpha \to x_0, Q(x_0) \ +B \subseteq \text{int}U \text{ for some neighborhood } B \text{ of } 0]$$
$$\Longrightarrow [\exists\bar{\alpha}, Q(x_{\bar{\alpha}}) \subseteq \text{int}U].$$

(v) Q is called lower semicontinuous with respect to U at x_0 if, $\forall x_\alpha \to x_0$, $\forall y \in Q(x_0)\backslash U$, $\exists y_\alpha \in Q(x_\alpha)$, $y_\alpha \to y$.

Similarly as for Definition 2.1 here int$U$ may be empty.

The following two examples explain that the properties of the above definitions are very easily satisfied. Moreover, the notions in Definition 2.1 are in a sense quite different from that in Definition 2.2.

Example 2.1 Let $X = Y = R$, $U = [0, 1]$ and $Q : R \to 2^R$ be defined by

$$Q(x) = \begin{cases} [\frac{1}{4}, \frac{1}{2}], & \text{if } \ x = 0, \\ [\frac{3}{2}, 2], & \text{if } \ x \neq 0. \end{cases}$$

Then Q satisfies all the four properties of Definition 2.1, but Q does not fulfilled any one among (i)–(iv) of Definition 2.2 (it satisfies (v)).

Example 2.2 Let $X = Y = R$, $U = R_+$ and $Q : R \to 2^R$ be defined by

$$Q(x) = \begin{cases} [-1, -\frac{1}{2}], & \text{if } \ x = 0, \\ (0, 2], & \text{if } \ x \neq 0. \end{cases}$$

Then, Q meets all conditions (i)–(iv) of Definition 2.2, but Q violates each property of Definition 2.1.

On the other hand, the following proposition shows that the above two definitions are closely related in another sense.

Proposition 2.1 *Let X, Y, Q, and U be as in Definition 2.1.*

(i) *Q is U-lsc at x_0 if and only if Q is $Y\backslash U$-lower-level closed at x_0.*
(ii) *Q is U-Hlsc at x_0 if and only if Q is $Y\backslash U$-Hausdorff-lower-level closed at x_0.*
(iii) *Q is U-usc at x_0 if and only if Q is $-U$-upper-level closed at x_0.*
(iv) *Q is U-Husc at x_0 if and only if Q is $-U$-Hausdorff-upper-level closed at x_0.*

Proof By the similarity we demonstrate only (i) and (iv).

(i) For the "only if" suppose Q is U-lsc at x_0 but there is $x_\alpha \to x_0$ such that $Q(x_\alpha) \subseteq \text{cl}(Y\backslash U) = Y\backslash\text{int}U$ but $Q(x_0) \not\subseteq Y\backslash\text{int}U$. Then $Q(x_0) \cap \text{int}U \neq \emptyset$. Since Q is U-lsc at x_0, there exists $\bar{\alpha}$ with $Q(x_{\bar{\alpha}}) \cap \text{int}U \neq \emptyset$, which is absurd.
For the "if" suppose Q is $Y\backslash U$-lower-level closed at x_0 but there exists $x_\alpha \to x_0$ such that $Q(x_0) \cap \text{int}U \neq \emptyset$ and $Q(x_\alpha) \cap \text{int}U = \emptyset$, $\forall\alpha$. Then $Q(x_\alpha) \subseteq Y\backslash\text{int}U = \text{cl}(Y\backslash U)$. Since Q is $Y\backslash U$-lower-level closed at x_0, the last inclusion implies a contradiction that $Q(x_0) \subseteq Y\backslash\text{int}U$.
(iv) For the "only if" suppose Q is U- Husc at x_0 but a net x_α tending to x_0 exists such that $Q(x_\alpha) \not\subseteq \text{int}U$, $\forall\alpha$, and there is a neighborhood B of 0 such that $Q(x_0) + B \subseteq \text{int}U$. As Q is U- Husc at x_0, the last inclusion implies that $\exists\bar{\alpha}$, $Q(x_{\bar{\alpha}}) \subseteq \text{int}U$, which is impossible.

For the "if" suppose Q is $-U$-Hausdorff-upper-level closed but there are $x_\alpha \to x_0$, a neighborhood B of x_0 such that $Q(x_0) + B \subseteq \text{int}U$ and $Q(x_\alpha) \not\subseteq \text{int}U$, $\forall\alpha$. Then, by the $-U$-Hausdorff-upper-level closedness, $Q(x_0) + B \not\subseteq \text{int}U$ for each neighborhood B of 0, a contradiction. □

The following assertion justifies the terminology used in Definition 2.1.

Proposition 2.2 *Let X, Y, Q, and U be as in Definition 2.1.*

(i) *Q is U-lower-level closed if and only if the lower-level set $\{x \mid Q(x) \subseteq \mathrm{cl}U\}$ is closed, if and only if Q is $Y \backslash U$-lsc.*
(ii) *Q is U-upper-level closed if and only if the upper-level set $\{x \mid Q(x) \not\subseteq -\mathrm{int}U\}$ is closed, if and only if Q is $-U$-usc.*
(iii) *Q is lsc at x_0 if and only if Q is U-lower-level closed at x_0 for each $U \subseteq Y$.*
(iv) *Q is Hlsc at x_0 if and only if Q is U-Hausdorff-lower-level closed at x_0 for each $U \subseteq Y$.*
(v) *Q is usc at x_0 if and only if Q is U-upper-level closed at x_0 for each $U \subseteq Y$.*
(vi) *Q is Husc at x_0 if and only if Q is U-Hausdorff-upper-level closed at x_0 for each $U \subseteq Y$.*

Proof (i) and (ii) are obvious.

(iii) "If". Suppose that Q is U-lower-level closed for each $U \subseteq Y$ but for some open subset V with $Q(x_0) \cap V \neq \emptyset$ there is $x_\alpha \to x_0$ with $Q(x_\alpha) \cap V = \emptyset$. Then $Q(x_\alpha) \subseteq Y \backslash V := U = \mathrm{cl}U$. By the U-lower-level closedness $Q(x_0) \subseteq \mathrm{cl}U = Y \backslash V$, i.e $Q(x_0) \cap V = \emptyset$, a contradiction.

"Only if." Suppose that Q is lsc at x_0 but there are $x_\alpha \to x_0$ and $U \subseteq Y$ with $Q(x_\alpha) \subseteq \mathrm{cl}U$, $\forall \alpha$, and $Q(x_0) \not\subseteq \mathrm{cl}U$, i.e., some $y_0 \in Q(x_0) \backslash \mathrm{cl}U$ exists. By the lower semicontinuity at x_0, there is $y_\alpha \in Q(x_\alpha)$, $y_\alpha \to y_0$. As $y_\alpha \in \mathrm{cl}U$, $y_0 \in \mathrm{cl}U$, which is impossible.

(iv)–(vi) It is checked similarly as (iii). □

The next proposition justifies the termins employed in Definition 2.2.

Proposition 2.3 *Let X, Y, Q and U be as in Definition 2.1*

(i) *$Q(.)$ is lsc at $x_0 \in X$ if and only if $Q(.) \backslash \mathrm{cl}U$ is lsc at x_0 for all $U \subseteq Y$.*
(ii) *$Q(.)$ is lsc at $x_0 \in X$ if and only if $Q(.)$ is lsc with respect to U at x_0 for all $U \subseteq Y$.*
(iii) *$Q(.)$ is usc at $x_0 \in X$ if and only if $Q(.) \backslash -\mathrm{int}U$ is usc at x_0 for all $U \subseteq Y$.*
(iv) *$Q(.)$ is Husc at x_0 if $Q(.) \backslash -\mathrm{int}U$ is Husc at x_0 for all $U \subseteq Y$. The converse is true if $Q(x_0)$ is compact.*

Proof (i) To check the "only if" let $y_0 \in Q(x_0) \backslash \mathrm{cl}U$ and $x_\alpha \to x_0$. Since $Q(.)$ is lsc at x_0, there is $y_\alpha \in Q(x_\alpha)$, $y_\alpha \to y_0$. Because $y_0 \notin \mathrm{cl}U$ we can assume that $y_\alpha \notin \mathrm{cl}U$, $\forall \alpha$, i.e., $y_\alpha \in Q(x_\alpha) \backslash \mathrm{cl}U$. This means the lower semicontinuity of $Q(.) \backslash \mathrm{cl}U$.

For the "if" suppose that Q is not lsc at x_0, i.e., $\exists y_0 \in Q(x_0), \exists x_\alpha \to x_0, \forall y_\alpha \in Q(x_\alpha)$, $y_\alpha \not\to y_0$. Take arbitrarily a closed subset U which does not contain y_0. Then any $y_\alpha \in Q(x_\alpha) \backslash \mathrm{cl}U \subseteq Q(x_\alpha)$ cannot tend to y_0. This contradicts the lower semicontinuity of $Q(.) \backslash \mathrm{cl}U$.

(ii) and (iii) are proved similarly.
(iv) For the "if" let U be such that $\mathrm{int}U = \emptyset$.

For the "converse", if $Q(x_0)$ is compact and $Q(.)$ is Husc, by Proposition 3.1 (Anh and Khanh 2004) $Q(.)$ is usc at x_0. Hence $Q(.) \backslash -\mathrm{int}U$ is usc at x_0 for all $U \subseteq Y$ by (iii). Due to (d) in Sect. 1, $Q(.) \backslash -\mathrm{int}U$ is Husc at x_0 for all $U \subseteq Y$. □

The following example shows that in (iv) the compactness of $Q(x_0)$ is essential.

Example 2.3 Let $X = Y = R$, $Q(x) = (x, x + 4)$, $x_0 = 0$, and $U = (-4, -2)$.

It is clear that $Q(.)$ is Husc at 0 ($Q(.)$ is not usc at 0), but $Q(.) \backslash -\mathrm{int}U$ is not Husc at 0. Indeed, let $x_n = \frac{1}{n}$ and $B = (-1, 1)$. Some direct computations show that $x_n \to 0$ and $Q(x_n) \backslash -\mathrm{int}U = (\frac{1}{n}, 2] \cup [4, \frac{1}{n} + 4) \not\subseteq Q(0) \backslash -\mathrm{int}U + B = (0, 2] + B = (-1, 3)$, $\forall n$. The reason is that $Q(0) = (0, 4)$ is not compact.

The following proposition is not hard to verify, gives additional relations and helps to have a clear insight.

Proposition 2.4 *Let X, Y, Q, and U be as in Definition 2.1.*

(i) *Q is lsc at x_0 if and only if Q is U-lsc at x_0 for all U.*
(ii) *Q is usc at x_0 if and only if Q is U-usc at x_0 for all U.*
(iii) *Q is U-lsc, U-usc, U-Hlsc or U-Husc at x_0 if and only if Q is* int$U$*-lsc,* int$U$*-usc,* int$U$*-Hlsc or* int$U$*-Husc at x_0, respectively.*
(iv) *U-Hausdorff-lower semicontinuity implies U-lower semicontinuity. The converse is not true even under compactness assumptions.*
(v) *U-upper semicontinuity implies U-Hausdorff-upper semicontinuity. If $Q(x_0)$ is compact then the converse is true at x_0.*
(vi) *Q is lsc with respect to U at x_0 if and only if Q is lsc with respect to V at x_0, for all $V \supseteq U$.*
(vii) *Q is lsc with respect to U at x_0 if $Q(.)\backslash U$ is lsc at x_0. The converse is true if U is closed.*

The following Examples 2.5 and 2.6 show that in (v) and (vii) we do not have the inverse implications without the respective compactness and closedness. Example 2.4 ensures that the converse of (iv) is not true even under the corresponding compactness assumption.

Example 2.4 Let X, Y, and x_0 be as in Example 2.3, and let $U = R_+$, $Q(0) = [0, 2]$ and $Q(x) = [0, 1]$ for $x \neq 0$. It is easy to see that $Q(.)$ is R_+-lsc at 0 and $Q(x)$ is compact $\forall x \in R$. But $Q(.)$ is not R_+-Hlsc at 0. Indeed, picking $B = (-\frac{1}{2}, \frac{1}{2})$ we see that $\forall x_\alpha \to 0$, $x_\alpha \neq 0$, $Q(0) \cap \mathrm{int}R_+ = (0, 2] \not\subseteq Q(x_\alpha) + B = (-\frac{1}{2}, \frac{3}{2})$, $\forall \alpha$.

Example 2.5 Let X, Y, Q, and x_0 be as in Example 2.3, and let $U = (0, 4)$. We easily see that $Q(.)$ is U-Husc at 0, but $Q(.)$ is not U-usc at 0, since $Q(0) \subseteq (0, 4)$ but, for $x_n = \frac{1}{n}$, $Q(x_n) \not\subseteq (0, 4)$, $\forall n$.

Example 2.6 Let X, Y, and x_0 be as in Example 2.3, $Q(x) = [|x|, |x| + 2]$ and $U = (0, 1]$. Then $Q(0)\backslash U = \{0\} \cup (1, 2]$ and $Q(x) = (1, |x| + 2]$, $\forall x \neq 0$. Hence $Q(.)\backslash U$ is not lsc at 0 but $Q(.)$ is lsc with respect to U. The reason is that U is not closed.

The following definition in Anh and Khanh (2004) is closely related to Definition 2.2.

Definition 2.3 Let X, Y, Q and U be as in Definition 2.1.

(i) Q is called to have the U-inclusion property at x_0 if $[x_\alpha \to x_0, Q(x_0) \cap (Y\backslash -\mathrm{int}U) \neq \emptyset] \Longrightarrow [\exists \bar{\alpha}, Q(x_{\bar{\alpha}}) \cap (Y\backslash -\mathrm{int}U) \neq \emptyset]$.
(ii) Q is said to have the strict U-inclusion property at x_0 if $[x_\alpha \to x_0, Q(x_0) \subseteq Y\backslash -\mathrm{int}U] \Longrightarrow [\exists \bar{\alpha}, Q(x_{\bar{\alpha}}) \subseteq Y\backslash -\mathrm{int}U]$.

Note that the difference between Definitions 2.2 and 2.3 is that the set int$U$ in the former is always open and $Y\backslash -\mathrm{int}U$ in the latter is always closed. An example of a mapping with both the U-inclusion and strict U-inclusion properties is F in Example 3.3 and F and G in Example 3.4.

3 Lower-semicontinuity-related results

In the sequel let $\mathrm{Sol}_1(\lambda, \mu)$ and $\mathrm{Sol}_2(\lambda, \mu)$ be the solution sets of (SQEP_1) and (SQEP_2), respectively, at (λ, μ) and let

$$E(\lambda) := \{(x, y) \mid x \in S(x, y, \lambda), y \in T(x, y, \lambda)\}.$$

Theorem 3.1 *Assume for problem $(SQEP_1)$ that, for $\emptyset \neq U \subseteq X \times Y$,*

(i_l) *$E(.)\backslash \mathrm{cl}U$ is lsc at λ_0;*
(ii_u) *S, T, A, and B are usc and compact valued in $K \times D \times \{\lambda_0\}$;*
($\mathrm{iii}_\mathrm{l}^\mathrm{l}$) *$F$ and G are $(Z\backslash -C)$-lsc in $K \times D \times K \times \{\mu_0\}$ and $D \times K \times D \times \{\mu_0\}$, respectively;*
(iv_1) *for each $(\bar{x}, \bar{y}) \in Sol_1(\lambda_0, \mu_0)$,*

$$F(x, \bar{y}, x^*, \mu_0) \cap (Z\backslash - C) \neq \emptyset, \forall x \in S(\bar{x}, \bar{y}, \lambda_0), \forall x^* \in A(\bar{x}, \bar{y}, \lambda_0),$$
$$G(y, \bar{x}, y^*, \mu_0) \cap (Z\backslash - C) \neq \emptyset, \forall y \in T(\bar{x}, \bar{y}, \lambda_0), \forall y^* \in B(\bar{x}, \bar{y}, \lambda_0).$$

Then $\mathrm{Sol}_1(.,.)$ *is U-lower-level closed at (λ_0, μ_0).*

Proof Arguing by contraposition, suppose the existence of $(\lambda_\alpha, \mu_\alpha) \to (\lambda_0, \mu_0)$ such that $\mathrm{Sol}_1(\lambda_\alpha, \mu_\alpha) \subseteq \mathrm{cl}U, \forall\alpha$, but $(x_0, y_0) \in \mathrm{Sol}_1(\lambda_0, \mu_0)\backslash \mathrm{cl}U$ exists. Then $\forall(x_\alpha, y_\alpha) \in \mathrm{Sol}_1(\lambda_\alpha, \mu_\alpha)$, $(x_\alpha, y_\alpha) \not\to (x_0, y_0)$. Since $E(.)\backslash \mathrm{cl}U$ is lsc at λ_0, there is $(\bar{x}_\alpha, \bar{y}_\alpha) \in E(\lambda_\alpha)\backslash \mathrm{cl}U$, $(\bar{x}_\alpha, \bar{y}_\alpha) \to (x_0, y_0)$. By the contradiction assumption, there exists a subnet $(\bar{x}_\beta, \bar{y}_\beta) \notin \mathrm{Sol}_1(\lambda_\beta, \mu_\beta), \forall\beta$. This means the existence of $\hat{x}_\beta \in S(\bar{x}_\beta, \bar{y}_\beta, \lambda_\beta)$, $\bar{x}^*_\beta \in A(\bar{x}_\beta, \bar{y}_\beta, \lambda_\beta)$,

$$F(\hat{x}_\beta, \bar{y}_\beta, \bar{x}^*_\beta, \mu_\beta) \subseteq -\mathrm{int}C, \tag{1}$$

or for some $\hat{y}_\beta \in T(\bar{x}_\beta, \bar{y}_\beta, \lambda_\beta)$, $\bar{y}^*_\beta \in B(\bar{x}_\beta, \bar{y}_\beta, \lambda_\beta)$,

$$G(\hat{y}_\beta, \bar{x}_\beta, \bar{y}^*_\beta, \mu_\beta) \subseteq -\mathrm{int}C. \tag{2}$$

Assume that (1) is fulfilled. Since S, A are usc at (x_0, y_0, λ_0) and $S(x_0, y_0, \lambda_0)$, $A(x_0, y_0, \lambda_0)$ are compact, one has $\hat{x}_0 \in S(x_0, y_0, \lambda_0)$, $\bar{x}^*_0 \in A(x_0, y_0, \lambda_0)$ such that $\hat{x}_\beta \to \hat{x}_0$, $\bar{x}^*_\beta \to \bar{x}^*_0$, (taking subnets if necessary). By (iv_1), we have

$$F(\hat{x}_0, y_0, \bar{x}^*_0, \mu_0) \cap (Z\backslash - C) \neq \emptyset. \tag{3}$$

By the $(Z\backslash{-C})$-lower semicontinuity of F at $(\hat{x}_0, y_0, \bar{x}^*_0, \mu_0)$, we see a contradiction between (1) and (3). If (2) holds, the reasoning is similar. □

To emphasize the symmetry and other relations between the assumptions of our theorems we adopt some subscripts and superscripts. A subscript l as in (i_l) means that this assumption is about lower semicontinuity. A superscript l as in ($\mathrm{iii}_\mathrm{l}^\mathrm{l}$) says that this assumption in imposed to get a lower semicontinuity result.

Taking into account Propositions 2.2 and 2.3 we obtain the following immediate consequence of Theorem 3.1.

Corollary 3.1 *Assume for problem $(SQEP_1)$ assumptions $(ii_u) - (iv_1)$ of Theorem 3.1. Assume further that*

(i'_l) E is lsc at λ_0.

Then $Sol_1(.,.)$ is lsc at (λ_0, μ_0).

If $X \equiv Y$, $K \equiv D$, then setting $S(x, y, \lambda) := S(y, \lambda)$, $T(x, y, \lambda) := \mathrm{cl}S(y, \lambda)$, $A(x, y, \lambda) := A(y, \lambda)$, $B(x, y, \lambda) \equiv K$, $F(x, \bar{x}, x^*, \mu) := F(x, x^*, \mu)$ and $G(y, \bar{x}, y^*, \mu) \equiv C$, our problems (SQEP$_1$) and (SQEP$_2$) collapse to problems ($\mathrm{P}_{s\alpha_1}$) and ($\mathrm{P}_{s\alpha_2}$), respectively, investigated in Anh and Khanh (Anh and Khanh (2007a)). The following example shows that in this case Corollary 3.1 improves Theorem 2.2 in Anh and Khanh (Anh and Khanh (2007a)).

Example 3.1 Let $X = Y = R$, $\Lambda \equiv M = [0, 1]$, $K = R$, $C = R_+$, $S(x, \lambda) = [0, 1]$, $A(x, \lambda) = \{x\}$, $\lambda_0 = 0$ and

$$F(x, x^*, \lambda) = \begin{cases} \{1\} & \text{if } \lambda = 0, \\ \{2\} & \text{otherwise}, \end{cases}$$

Then all assumptions of Corollary 3.1 are fulfilled. By this corollary the solution set is lsc at 0 (in fact $\mathrm{Sol}_1(\lambda) = [0, 1]$, $\forall \lambda \in [0, 1]$), but Theorem 2.2 in Anh and Khanh (2007a) cannot be applied since F is not lsc at 0.

Furthermore, if in addition, $A(x, \lambda) = \{x\}$ then our problems become (QEP) and (SQEP), respectively, studied in Anh and Khanh (2004). Example 3.1 shows also that Corollary 3.1 is strictly stronger Theorem 2.1 in Anh and Khanh (2004).

The following example shows that the rather strong and oddly looking assumption (iv$_1$) cannot be dropped.

Example 3.2 Let $X = Y = Z = R$, $\Lambda \equiv M = [0, 1]$, $C = R_+$, $S(x, y, \lambda) = T(x, y, \lambda) = A(x, y, \lambda) = B(x, y, \lambda) = [0, 1]$, $F(x, y, x^*, \lambda) = \{\lambda(y - x)\}$, $G(y, x, y^*, \lambda) = \{1\}$ and $\lambda_0 = 0$. Then $(\mathrm{i}_1') - (\mathrm{iii}_1^1)$ are clearly satisfied. However, some direct computation gives $\mathrm{Sol}_1(0) = [0, 1]$ and $\mathrm{Sol}_1(\lambda) = \{1\}$ for each $\lambda > 0$ and hence $\mathrm{Sol}_1(.)$ is not lsc at 0. The reason is that (iv$_1$) is violated.

Although assumption (iv$_1$) is essential, it together with (iii_l^l) can be replaced by a condition relating F and G as follows.

Theorem 3.2 *Assume* (i_l) *and* (ii_u) *as in Theorem 3.1 and replace* (iii_l^l) *and* (iv_1) *by*

(iii$_1$) *F and G have the C-inclusion property in* $K \times D \times K \times \{\mu_0\}$ *and* $D \times K \times D \times \{\mu_0\}$*, respectively.*

Then $Sol_1(., .)$ *is U-lower-level closed at* (λ_0, μ_0).

Proof The first part of the proof of Theorem 3.1 (until the last sentence before (3)), using only (i_l) and (ii_u) remains valid here. Now assumption (iii$_1$) together with the fact that $(x_0, y_0) \in \mathrm{Sol}_1(\lambda_0, \mu_0)$ imply the existence of β_1, β_2 such that $F(\hat{x}_{\beta_1}, \bar{y}_{\beta_1}, \bar{x}^*_{\beta_1}, \mu_{\beta_1}) \cap (Z \backslash -\mathrm{int}C) \neq \emptyset$ and $G(\hat{y}_{\beta_2}, \bar{x}_{\beta_2}, \bar{y}^*_{\beta_2}, \mu_{\beta_2}) \cap (Z \backslash -\mathrm{int}C) \neq \emptyset$, which contradicts (1) or (2), respectively. □

We clearly have a direct consequence as follows.

Corollary 3.2 *Assume* (ii_u) *and* (iii_1) *as in Theorem 3.2 and replace* (i_l) *by*

(i_l') *E is lsc at* λ_0.

Then $\mathrm{Sol}_1(., .)$ *is lsc at* (λ_0, μ_0).

When the symmetric quasiequilibrium problems are particularized as quasiequilibrium problems, Corollary 3.2 coincides with Theorem 2.2 in Anh and Khanh (2004). The following example shows that the assumptions of this corollary are easier to check than that of Theorem 2.1 in Anh and Khanh (2007a).

Example 3.3 Let $X, Y, \Lambda, M, K, C, \lambda_0$ be as in Example 3.1 and $S(x,\lambda) = [\lambda, \lambda+1]$, $A(x, \lambda) = [\sin\alpha, 2]$ and

$$F(x, x^*, \lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0, \\ \{1\} & \text{otherwise.} \end{cases}$$

It is easy to see that all assumptions of Corollary 3.2 are fulfilled but it is difficult to verify the openness of $U_{r\alpha}$ in Theorem 2.1 of Anh and Khanh (2007a).

The main advantage of assumption (iii$_1$) is that it does not require any information on the solution set $\mathrm{Sol}_1(\lambda_0, \mu_0)$. Moreover, (iii$_1$) may be satisfied even in cases, where both (iii$_1^l$) and (iv$_1$) are not fulfilled as shown by the following example.

Example 3.4 Let $X = Y = Z = R$, $\Lambda \equiv M = [0,1]$, $K = D = R$, $C = R_+$, $S(x, y, \lambda) = T(x, y, \lambda) = A(x, y, \lambda) = B(x, y, \lambda) = [0, 1]$, $\lambda_0 = 0$ and

$$F(x, y, \hat{x}, \lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0, \\ \{1\} & \text{otherwise,} \end{cases}$$

$$G(y, x, \hat{y}, \lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0, \\ \{\frac{1}{2}\} & \text{otherwise.} \end{cases}$$

Then, it is not hard to see that (i$_l$), (ii$_u$) and (iii$_1$) are satisfied and, according to Theorem 3.2, $\mathrm{Sol}_1(.)$ is lsc at 0 (in fact $\mathrm{Sol}_1(\lambda) = [0, 1]$, for all $\lambda \in [0, 1]$). Evidently (iii$_1^l$) and (iv$_1$) are not fulfilled in this case.

The following example shows that the assumption (i$_1'$) is essential.

Example 3.5 Let $X = Y = Z = R$, $\Lambda \equiv M = [0, 1]$, $K = D = R$, $C = R_+$, $\lambda_0 = 0$, $A(x, y, \lambda) = \{x\}$, $B(x, y, \lambda) = \{y\}$ and

$$S(x, y, \lambda) = \begin{cases} [-1, 1] & \text{if } \lambda = 0, \\ [-\lambda - 1, 0] & \text{if } \lambda \neq 0, \end{cases}$$

$$T(x, y, \lambda) \equiv \{1\},$$
$$F(x, y, x^*, \lambda) = G(y, x, y^*, \lambda) \equiv \{1\}.$$

Then all assumptions but (i$_1'$) of Corollaries 3.1 and 3.2 are satisfied. In fact, $E(0) = [-1, 1] \times \{1\}$ and $E(\lambda) = [-\lambda - 1, 0] \times \{1\}, \forall \lambda \neq 0$. So E is not lsc at $\lambda_0 = 0$. However for $U = C \times C$, $E(0)\backslash \mathrm{cl}U = [-1, 0) \times \{1\}$ and $E(\lambda)\backslash \mathrm{cl}U = [-\lambda - 1, 0) \times \{1\}$ for $\lambda \neq 0$ and hence $E(.)\backslash \mathrm{cl}U$ is lsc at λ_0. Checking directly we see that $\mathrm{Sol}_1(0) = [-1, 1] \times \{1\}$ and $\mathrm{Sol}_1(\lambda) = [-\lambda - 1, 0] \times \{1\}$ for $\lambda \neq 0$. Then $\mathrm{Sol}_1(.)$ is U-lower-level closed at λ_0 but $\mathrm{Sol}_1(.)$ is not lsc at λ_0.

Passing to problem (SQEP$_2$) we easily get the following corresponding results, which are given without proofs.

Theorem 3.3 *Assume for problem* $(SQEP_2)$, (i_l) *and* (ii_u) *of Theorem 3.1. Assume further that*

(iii_u^l) *F and G are* $(Z\backslash -C)$*-usc in* $K \times D \times K \times \{\mu_0\}$ *and* $D \times K \times D \times \{\mu_0\}$*, respectively;*

(iv$_2$) for each $(\bar{x}, \bar{y}) \in Sol_2(\lambda_0, \mu_0)$,

$$F(x, \bar{y}, x^*, \mu_0) \subseteq Z \backslash -C, \forall x \in S(\bar{x}, \bar{y}, \lambda_0), \forall x^* \in A(\bar{x}, \bar{y}, \lambda_0),$$
$$G(y, \bar{x}, y^*, \mu_0) \subseteq Z \backslash -C, \forall y \in T(\bar{x}, \bar{y}, \lambda_0), \forall y^* \in B(\bar{x}, \bar{y}, \lambda_0).$$

Then $Sol_2(.,.)$ *is* U*-lower-level closed at* (λ_0, μ_0).

Corollary 3.3 *Assume* (ii_u), (iii_u^l) *and* (iv_2) *as in Theorem 3.3 and replace* (i_l) *by*

(i_l') *E is lsc at* λ_0.

Then $Sol_2(.,.)$ *is lsc at* (λ_0, μ_0).

Example 3.1 shows also that Corollary 3.3 strictly includes Theorem 2.3 in Anh and Khanh (2004) and Theorem 2.2 in Anh and Khanh (2007a).

Theorem 3.4 *Assume for problem* $(SQEP_2)$, (i_l), *and* (ii_u). *Assume further that*

(iii$_2$) F and G have the strict C-inclusion property in $K \times D \times K \times \{\mu_0\}$ *and* $D \times K \times D \times \{\mu_0\}$, *respectively.*

Then $Sol_2(.,.)$ *is* U*-lower-level closed at* (λ_0, μ_0).

Corollary 3.4 *Assume* (ii_u) *and* (iii_2) *as in Theorem 3.4 and replace* (i_l) *by*

(i_l') *E is lsc at* λ_0.

Then $Sol_2(.,.)$ *is lsc at* (λ_0, μ_0).

Corollary 3.4 coincides with Theorem 2.4 in Anh and Khanh (2004). In comparison with the corresponding result of Anh and Khanh (2007a), Example 3.3 gives a case where the assumptions of this corollary are easier to be checked (than that of Theorem 2.1 in Anh and Khanh (2007a)).

Example 3.4 indicates also that (iii$_2$) may be satisfied even when both (iii$_\mathrm{u}^\mathrm{l}$) and (iv$_2$) are violated, since here F and G are single-valued and (iii$_1$) coincides with (iii$_2$).

We now proceed to Hausdorff lower semicontinuity.

Theorem 3.5 *Assume for* ($SQEP_1$) (ii_u), (iii_l^l), *and* (iv_1) *of Theorem 3.1. Assume further, for* $\emptyset \neq U \subseteq X \times Y$, *that*

(i) E is lsc with respect to int$U$ *at* λ_0, $E(\lambda_0)\backslash \text{int}U$ *is compact and* $S(.,.,\lambda_0)$ *is closed in* $K \times D \times \{\lambda_0\}$;
(ii) $S(.,.,\lambda_0)$, $T(.,.,\lambda_0)$, $A(.,.,\lambda_0)$ *and* $B(.,.,\lambda_0)$ *are lsc;*
(iii) $F(.,.,\mu_0)$ *and* $G(.,.,\mu_0)$ *are* $-C$*-usc in* $K \times D \times K$ *and* $D \times K \times D$, *respectively.*

Then $Sol_1(.,.)$ *is* U*-Hausdorff-lower-level closed at* (λ_0, μ_0).

Proof We first show that $\text{Sol}_1(\lambda_0, \mu_0)$ is closed in $X \times Y$. Suppose that $(x_\alpha, y_\alpha) \in \text{Sol}_1(\lambda_0, \mu_0)$, $(x_\alpha, y_\alpha) \to (x_0, y_0)$. If $(x_0, y_0) \notin \text{Sol}_1(\lambda_0, \mu_0)$. Then there exist $\hat{x}_0 \in S(x_0, y_0, \lambda_0)$, $x_0^* \in A(x_0, y_0, \lambda_0)$,

$$F(\hat{x}_0, y_0, x_0^*, \mu_0) \subseteq -\text{int}C, \tag{4}$$

or $\hat{y}_0 \in T(x_0, y_0, \lambda_0)$, $y_0^* \in B(x_0, y_0, \lambda_0)$,

$$G(\hat{y}_0, x_0, y_0^*, \mu_0) \subseteq -\text{int}C. \tag{5}$$

Suppose (4) is fulfilled. Since $S(.,.,\lambda_0)$ and $A(.,.,\lambda_0)$ are lsc in $K \times D$, there are $\hat{x}_\alpha \in S(x_\alpha, y_\alpha, \lambda_0)$, $x^*_\alpha \in A(x_\alpha, y_\alpha, \lambda_0)$ such that $(\hat{x}_\alpha, x^*_\alpha) \to (\hat{x}_0, x^*_0)$. As $(x_\alpha, y_\alpha) \in \mathrm{Sol}_1(\lambda_0, \mu_0)$, we have

$$F(\hat{x}_\alpha, y_\alpha, x^*_\alpha, \mu_0) \nsubseteq -\mathrm{int}C. \tag{6}$$

By the $-C$-upper semicontinuity of $F(.,.,.,\mu_0)$ in $K \times D \times K$, we see a contradiction between (4) and (6). The argument for the case, where (5) holds, is similar. Hence, $\mathrm{Sol}_1(\lambda_0, \mu_0)$ is closed and hence $\mathrm{Sol}_1(\lambda_0, \mu_0)\setminus\mathrm{int}U$ is compact, by (i).

We show that $\forall(\lambda_\alpha, \mu_\alpha) \to (\lambda_0, \mu_0), \forall(\bar{x}_0, \bar{y}_0) \in \mathrm{Sol}_1(\lambda_0, \mu_0) \setminus \mathrm{int}U, \exists(\bar{x}_\alpha, \bar{y}_\alpha) \in \mathrm{Sol}_1(\lambda_\alpha, \mu_\alpha), (\bar{x}_\alpha, \bar{y}_\alpha) \to (\bar{x}_0, \bar{y}_0)$. Suppose to the contrary that there exist $(\lambda_\alpha, \mu_\alpha) \to (\lambda_0, \mu_0)$ and $(\bar{x}_0, \bar{y}_0) \in \mathrm{Sol}_1(\lambda_0, \mu_0) \cap \big((X \times Y)\setminus\mathrm{int}U\big)$ such that $\forall(x_\alpha, y_\alpha) \in \mathrm{Sol}_1(\lambda_\alpha, \mu_\alpha)$, $(x_\alpha, y_\alpha) \nrightarrow (\bar{x}_0, \bar{y}_0)$. Since E is lsc with respect to $\mathrm{int}U$ at λ_0, there is $(\bar{x}_\alpha, \bar{y}_\alpha) \in E(\lambda_\alpha)$, $(\bar{x}_\alpha, \bar{y}_\alpha) \to (\bar{x}_0, \bar{y}_0)$. By the contradiction assumption, there exists a subnet $(\bar{x}_\beta, \bar{y}_\beta) \notin \mathrm{Sol}_1(\lambda_\beta, \mu_\beta), \forall\beta$. The further argument to see a contradiction is similar as that of Theorem 3.1.

Now suppose that $\mathrm{Sol}_1(.,.)$ is not U-Hausdorff-lower-level closed at (λ_0, μ_0), i.e., $\exists B$ (a neighborhood of the origin in $X \times Y$), $\exists(\lambda_\alpha, \mu_\alpha) \to (\lambda_0, \mu_0)$ such that $\forall\alpha, \exists(x_{0\alpha}, y_{0\alpha}) \in \mathrm{Sol}_1(\lambda_0, \mu_0) \setminus \mathrm{cl}U$, $(x_{0\alpha}, y_{0\alpha}) \notin \mathrm{Sol}_1(\lambda_\alpha, \mu_\alpha) + B$. Since $\mathrm{Sol}_1(\lambda_0, \mu_0) \setminus \mathrm{int}U$ is compact, we can assume that $(x_{0\alpha}, y_{0\alpha}) \to (x_0, y_0) \in \mathrm{Sol}_1(\lambda_0, \mu_0) \setminus \mathrm{int}U$. So we can suppose that there are α_1, a neighborhood B_1 of 0 in $X \times Y$ with $B_1 + B_1 \subseteq B$ and $b_\alpha \in B_1$ such that, $\forall\alpha \geq \alpha_1, (x_{0\alpha}, y_{0\alpha}) = (x_0, y_0) + b_\alpha$. By the preceding part of the proof there is $(x_\alpha, y_\alpha) \in \mathrm{Sol}_1(\lambda_\alpha, \mu_\alpha), (x_\alpha, y_\alpha) \to (x_0, y_0)$ and hence, one can assume that there is $\alpha_2, \forall\alpha \geq \alpha_2$,

$$(x_\alpha, y_\alpha) \in (x_0, y_0) - B_1,$$

i.e., there exists $b'_\alpha \in B_1, (x_\alpha, y_\alpha) = (x_0, y_0) - b'_\alpha$. Hence $\forall\alpha \geq \alpha_0 = \max\{\alpha_1, \alpha_2\}$,

$$(x_{0\alpha}, y_{0\alpha}) = (x_0, y_0) + b_\alpha = (x_\alpha, y_\alpha) + b'_\alpha + b_\alpha \in (x_\alpha, y_\alpha) + B.$$

This is impossible due to the fact that $(x_{0\alpha}, y_{0\alpha}) \notin \mathrm{Sol}_1(\lambda_\alpha, \mu_\alpha) + B$. Thus, $\mathrm{Sol}_1(.,.)$ is U-Hausdorff-lower-level closed at (λ_0, μ_0). □

Propositions 2.2, 2.3 and Theorem 3.5 derive the following result.

Corollary 3.5 *Assume all assumptions as in Corollary 3.1 and Theorem 3.5 but* (i)*, and replace* (i) *by*

(i′) E is lsc at λ_0 and $E(\lambda_0)$ is compact.

Then $Sol_1(.,.)$ is Hlsc at $(\lambda_0.\mu_0)$.

The following example explains the essentialness of the compactness of $E(\lambda_0)$.

Example 3.6 Let $X = Y = Z = R$, $\Lambda \equiv M = [0, 1]$, $K = D = R$, $C = R_+$, $S(x, y, \lambda) = A(x, y, \lambda) = \{x\}$, $T(x, y, \lambda) = \{\lambda x\}$, $B(x, y, \lambda) = \{y\}$, $F(x, y, x^*, \mu) = G(y, x, y^*, \mu) \equiv \{1\}$.

It is clear that $E(\lambda) = \{(x, \lambda x) \in R^2 \mid x \in R\}$. So, E is lsc; S, T, A, and B are continuous and have compact values in $K \times D \times \Lambda$; F and G are continuous and compact valued in R^4. Hence, all assumptions of Corollary 3.5 but (i) are fulfilled. It is easy to see that $\mathrm{Sol}_1(\lambda) = E(\lambda) = \{(x, \lambda x) \mid x \in R\}$. Thus, $\mathrm{Sol}_1(.)$ is lsc in R. But $\forall\lambda_0 \in \Lambda$, $\mathrm{Sol}_1(.)$ is not Hlsc at λ_0, since $\forall\lambda \neq \lambda'$, $H\big(\mathrm{Sol}_1(\lambda), \mathrm{Sol}_1(\lambda')\big) = +\infty$, where $H(.,.)$ is the Hausdorff distance. The reason is that $E(\lambda_0)$ is not compact.

Similarly, we obtain the following results corresponding to Theorems 3.2–3.4 and Corollaries 3.2–3.4.

Theorem 3.6 *Assume all assumptions of Theorem 3.5 but* (iii_l^l) *and* (iv_1) *. Assume further that*

(iii_1) *F and G have the C-inclusion property in* $K \times D \times K \times \{\mu_0\}$ *and* $D \times K \times D \times \{\mu_0\}$*, respectively.*

Then $Sol_1(.,.)$ *is U-Hausdorff-lower-level closed at* (λ_0, μ_0)*.*

Corollary 3.6 *Assume all assumptions of Theorem 3.6 but* (i)*, and replace* (i) *by*

(i') *E is lsc at* λ_0 *and* $E(\lambda_0)$ *is compact.*

Then $Sol_1(.,.)$ *is Hlsc at* $(\lambda_0.\mu_0)$*.*

Theorem 3.7 *Assume all assumptions of Theorem 3.3 and* (i), (ii) *of Theorem 3.5. Assume further that*

(iii') $F(.,.,.,\mu_0)$ *and* $G(.,.,.,\mu_0)$ *are* $-C$*-lsc.*

Then $Sol_2(.,.)$ *is U-Hausdorff-lower-level closed at* (λ_0, μ_0)*.*

Corollary 3.7 *Assume all assumptions of Theorem 3.7 but* (i) *and replace* (i) *by*

(i') *E is lsc at* λ_0 *and* $E(\lambda_0)$ *is compact.*

Then $Sol_2(.,.)$ *is Hlsc at* $(\lambda_0.\mu_0)$*.*

Theorem 3.8 *Assume all assumptions of Theorem 3.7 but* (iii_{u^l}) *and* (iv_2)*. Assume further that*

(iii_2) *F and G have the strict C-inclusion property in* $K \times D \times K \times \{\mu_0\}$ *and* $D \times K \times D \times \{\mu_0\}$*, respectively.*

Then $Sol_2(.,.)$ *is U-Hausdorff-lower-level closed at* (λ_0, μ_0)*.*

Corollary 3.8 *Assume all assumptions of Theorem 3.8 but* (i) *and replace* (i) *by*

(i') *E is lsc at* λ_0 *and* $E(\lambda_0)$ *is compact.*

Then $Sol_2(.,.)$ *is Hlsc at* $(\lambda_0.\mu_0)$*.*

Example 3.6 shows also that the assumed compactness of $E(\lambda_0)$ is essential for Corollaries 3.6–3.8, since the C-inclusion properties are satisfied and F and G are single-valued.

4 Upper-semicontinuity-related results

Theorem 4.1 *Assume for problem* $(SQEP_1)$ *that, for* $U \subseteq X \times Y$*,*

- (i_u) $E(.)\backslash -\mathrm{int}U$ *is usc and* $E(\lambda_0)\backslash -\mathrm{int}U$ *is compact*;
- (ii_1) *S, T, A, and B are lsc in* $K \times D \times \{\lambda_0\}$;
- (iii_u^u) *F and G are* $(-C)$*-usc in* $K \times D \times K \times \{\mu_0\}$ *and* $D \times K \times D \times \{\mu_0\}$*, respectively.*

Then $Sol_1(.,.)$ *is U-upper-level closed at* (λ_0, μ_0)*.*

Proof Reasoning ab absurdo suppose the existence of $(\lambda_\alpha, \mu_\alpha) \to (\lambda_0, \mu_0)$ such that $\mathrm{Sol}_1(\lambda_\alpha, \mu_\alpha) \not\subseteq -\mathrm{int}U$ for all α but $\mathrm{Sol}_1(\lambda_0, \mu_0) \subseteq -\mathrm{int}U$. Then there exists $(x_\alpha, y_\alpha) \in \mathrm{Sol}_1(\lambda_\alpha, \mu_\alpha) \backslash -\mathrm{int}U$. By $(\mathrm{i_u})$ one can assume that (x_α, y_α) tends to some $(x_0, y_0) \in E(\lambda_0) \backslash -\mathrm{int}U$. If $(x_0, y_0) \notin \mathrm{Sol}_1(\lambda_0, \mu_0)$ then there are $\hat{x}_0 \in S(x_0, y_0, \lambda_0)$, $x_0^* \in A(x_0, y_0, \lambda_0)$,

$$F(\hat{x}_0, y_0, x_0^*, \mu_0) \subseteq -\mathrm{int}C, \tag{7}$$

or for some $\hat{y}_0 \in T(x_0, y_0, \lambda_0)$, $y_0^* \in B(x_0, y_0, \lambda_0)$,

$$G(\hat{y}_0, x_0, y_0^*, \mu_0) \subseteq -\mathrm{int}C. \tag{8}$$

If (7) is fulfilled, then since S and A are lsc at (x_0, y_0, λ_0), there exist $\hat{x}_\alpha \in S(x_\alpha, y_\alpha, \lambda_\alpha)$, $x_\alpha^* \in A(x_\alpha, y_\alpha, \lambda_\alpha)$ such that $\hat{x}_\alpha \to \hat{x}_0$, $x_\alpha^* \to x_0^*$. As F is $(-C)$-usc at $(\hat{x}_0, y_0, x_0^*, \mu_0)$ there must be then an $\bar{\alpha}$ such that $F(\hat{x}_{\bar{\alpha}}, y_{\bar{\alpha}}, x_{\bar{\alpha}}^*, \mu_{\bar{\alpha}}) \subseteq -\mathrm{int}C$, which is impossible as $(x_{\bar{\alpha}}, y_{\bar{\alpha}}) \in \mathrm{Sol}_1(\lambda_{\bar{\alpha}}, \mu_{\bar{\alpha}})$. If (8) holds one gets a similar contradiction. Thus $(x_0, y_0) \in \mathrm{Sol}_1(\lambda_0, \mu_0) \subseteq -\mathrm{int}U$, which contradicts the fact that $(x_\alpha, y_\alpha) \notin -\mathrm{int}U$ for all α. □

Corollary 4.1 *Assume* (ii_l) *and* (iii_u^u) *as in Theorem 4.1 and replace* (i_u) *by*

(i_u') *E is usc and* $E(\lambda_0)$ *is compact.*

Then $\mathrm{Sol}_1(.,.)$ *is both usc and closed at* (λ_0, μ_0).

Proof The upper semicontinuity follows immediately from Theorem 4.1 and Propositions 2.2 and 2.3.

Suppose that $\mathrm{Sol}_1(.,.)$ is not closed at (λ_0, μ_0), i.e., there is a net $(\lambda_\alpha, \mu_\alpha, x_\alpha, y_\alpha) \to (\lambda_0, \mu_0, x_0, y_0)$ with $(x_\alpha, y_\alpha) \in \mathrm{Sol}_1(\lambda_\alpha, \mu_\alpha)$ but $(x_0, y_0) \notin \mathrm{Sol}_1(\lambda_0, \mu_0)$. Then we repeat the second part of the proof of Theorem 4.1 to get a contradiction. □

In the case where our problems are reduced to quasiequilibrium problems investigated in Anh and Khanh (2004) and Anh and Khanh (2007a), Corollary 4.1 improves Theorem 3.1 in Anh and Khanh (2004), Theorems 3.1 and 4.1 in Bianchi and Pini (2003), while this corollary is weaker than Theorem 3.1 in Anh and Khanh (2007a) (but this corollary is easier to use). Example 3.1 shows also that this corollary is strictly stronger than Theorem 3.1 in Anh and Khanh (2004), since F is a single-valued function. The following example ensures that Corollary 4.1 improves the corresponding results in Bianchi and Pini (2003).

Example 4.1 Let $X = Z = R$, $\Lambda \equiv M = R$, $K = [0, 1]$, $C = R_+$, $S(x, \lambda) = K$, $A(x, \lambda)$ $\{x\}$, $\lambda_0 = 0$ and

$$F(x, x^*, \lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0, \\ \{1\} & \text{otherwise.} \end{cases}$$

Then all assumptions of Corollary 4.1 are fulfilled. Hence, this corollary yields the upper semicontinuity of the solution set, but Theorems 3.1 and 4.1 in Bianchi and Pini (2003) do not work, since F is neither pseudomonotone nor α-upper-level closed for all $\alpha > 0$.

Similarly one can obtain the same properties for problem (SQEP_2) as follows.

Theorem 4.2 *Assume for problem* $(SQEP_2)$ (i_u)*, and* (ii_l) *as in Theorem 4.1. Assume further that*

(iii_l^u) *F and G are* $(-C)$*-lsc in* $K \times D \times K \times \{\mu_0\}$ *and* $D \times K \times D \times \{\mu_0\}$*, respectively.*

Then $Sol_2(.,.)$ *is U-upper-level closed at* (λ_0, μ_0).

Corollary 4.2 *Assume* (ii_l) *and* (iii_l^u) *as in Theorem 4.2 and replace* (i_u) *by*

(i'_u) *E is usc and* $E(\lambda_0)$ *is compact.*

Then $Sol_2(.,.)$ *is both usc and closed at* (λ_0, μ_0).

For the special case of quasiequilibrium problems Corollary 4.2 is weaker than Theorem 3.1 in Anh and Khanh (2007a). Example 3.1 explains that it improves Theorem 3.4 in Anh and Khanh (2004).

The following example shows that assumption (i'_u) in Corollaries 4.1 and 4.2 is essential.

Example 4.2 Let $X{=}Y{=}Z{=}R$, $\Lambda \equiv M{=}[0, 1]$, $K{=}D{=}R$, $C{=}R_+$, $\lambda_0{=}0$, $A(x, y, \lambda){=}\{x\}$, $B(x, y, \lambda){=}\{y\}$ and

$$\begin{aligned}
S(x, y, \lambda) &= (-\lambda - 1, \lambda],\\
T(x, y, \lambda) &\equiv \{-1\},\\
F(x, y, x^*, \lambda) &= G(y, x, y^*, \lambda) \equiv \{1\}.
\end{aligned}$$

Then it is easy to see that all assumptions but (i'_u) of Corollaries 4.1 and 4.2 are fulfilled. For (i'_u) we check directly that $E(\lambda) = (-\lambda - 1, \lambda] \times \{1\}$ is not compact at $\lambda_0 = 0$, but for $U = C \times C$, $E(\lambda) \setminus -\mathrm{int}U = [0, \lambda] \times \{1\}$ and hence $E(\lambda_0) \setminus -\mathrm{int}U$ is compact and $E(.) \setminus -\mathrm{int}U$ is usc. By direct computation we get $\mathrm{Sol}_1(\lambda) = (-\lambda - 1, \lambda] \times \{-1\}$, which is neither usc nor closed at $\lambda_0 = 0$, although $\mathrm{Sol}_1(.)$ is U-upper-level closed at λ_0.

As mentioned in Sect. 1, our problems (SQEP_1) and (SQEP_2) include a wide range of optimization-related problems (see e.g., Anh and Khanh (2007b), Anh and Khanh (2007c) for more details about this inclusion). Hence one can derive consequences for these problems from the results here. In Sect. 6 we discuss corollaries only for one among these problems. In the example below we show how the typical classical problem of minimizing a (numerical) function satisfies the assumptions of Corollaries 4.1 and 4.2, to ensure the applicability of our general results.

Example 4.3 Let X be a Hausdorff topological vector space, M be a topological space, $K \subseteq X$ be compact and $\varphi : K \times M \to R$ be a function. Consider, for $\mu \in M$, the problem

$$(\mathrm{MP}) \qquad \min_{x \in K} \varphi(x, \mu).$$

Set $Y \equiv X$, $\Lambda = M$, $Z = R$, $C = R_+$, $D \equiv K$, $S(x, y, \lambda) = T(x, y, \lambda) \equiv K$, $A(x, y, \lambda) = B(x, y, \lambda) = \{x\}$, $F(x, y, x^*, \mu) = \varphi(x, \mu) - \varphi(x^*, \mu)$ and $G(y, x, y^*, \mu) = \{1\}$. Then, ($\mathrm{SQEP}_1$) and ($\mathrm{SQEP}_2$) become (MP). It is easy to see that assumptions (i'_u), (ii_l) of Corollaries 4.1 and 4.2 are satisfied and G fulfils $(\mathrm{iii}_\mathrm{u}^\mathrm{u})$ and $(\mathrm{iii}_\mathrm{l}^\mathrm{u})$ (they coincide in this case). This assumption for F, i.e., F is $-R_+$-usc in $K \times K \times \{\mu_0\}$, means that

$$\begin{aligned}
&[\exists (x_\alpha, x_\alpha^*, \mu_\alpha) \to (x_0, x_0^*, \mu_0), \forall \alpha, \varphi(x_\alpha, \mu_\alpha) - \varphi(x_\alpha^*, \mu_\alpha) \geq 0]\\
&\quad \Rightarrow [\varphi(x_0, \mu_0) - \varphi(x_0^*, \mu_0) \geq 0].
\end{aligned}$$

This in turn is equivalent to saying that the function $(x, x^*, \mu) \to \varphi(x, \mu) - \varphi(x^*, \mu)$ is R_+-upper-level closed in $K \times K \times \{\mu_0\}$. So for (MP), Corollaries 4.1 and 4.2 say that if the function $(x, x^*, \mu) \to \varphi(x, \mu) - \varphi(x^*, \mu)$ is R_+-upper-level closed in $K \times K \times \{\mu_0\}$, then the solution set of (MP) is usc at μ_0. (In this case the upper semicontinuity coincides with the closedness.)

Passing to Hausdorff upper-level closedness we see that the assumptions can be weakened correspondingly as follows.

Theorem 4.3 *Assume for problem (SQEP$_1$) that, for $\emptyset \neq U \subseteq X \times Y$,*

(i_{hu}) *$E(.)\backslash -\mathrm{int}U$ is Husc and $E(\lambda_0)\backslash -\mathrm{int}U$ is compact;*

(ii_l) *S, T, A and B are lsc in $K \times D \times \{\lambda_0\}$;*

(iii_{hu}) *F and G are $-C$-Husc in $K \times D \times K \times \{\mu_0\}$ and $D \times K \times D \times \{\mu_0\}$, respectively;*

(iv_h) *$\forall B$(open neighborhood of 0 in $X \times Y$), $\forall (x, y) \notin S_1(\lambda_0, \mu_0) + B$, $\exists B_Z$ (neighborhood of 0 in Z), $\exists \hat{x} \in S(x, y, \lambda_0)$, $\exists x^* \in A(x, y, \lambda_0)$ such that*

$$F(\hat{x}, y, x^*, \mu_0) + B_Z \subseteq -\mathrm{int}C,$$

or $\exists \hat{y} \in T(x, y, \lambda_0)$, $\exists y^* \in B(x, y, \lambda_0)$ such that

$$G(\hat{y}, x, y^*, \mu_0) + B_Z \subseteq -\mathrm{int}C.$$

Then $Sol_1(., .)$ is U-Hausdorff-upper-level closed at (λ_0, μ_0).

Proof Suppose to the contrary that there are a net $(\lambda_\alpha, \mu_\alpha) \to (\lambda_0, \mu_0)$ and an open neighborhood B of 0 in $X \times Y$ such that $\mathrm{Sol}_1(\lambda_\alpha, \mu_\alpha) \nsubseteq -\mathrm{int}U$ for all α but $\mathrm{Sol}_1(\lambda_0, \mu_0) + B \subseteq -\mathrm{int}U$. There exists then $(x_\alpha, y_\alpha) \in \mathrm{Sol}_1(\lambda_\alpha, \mu_\alpha)\backslash -\mathrm{int}U$. By the compactness of $E(\lambda_0)\backslash -\mathrm{int}U$ and the Hausdorff upper semicontinuity of $E(.)\backslash -\mathrm{int}U$ at λ_0, we can assume that $(x_\alpha, y_\alpha) \to (x_0, y_0)$ for some $(x_0, y_0) \in E(\lambda_0)\backslash -\mathrm{int}U$. If $(x_0, y_0) \notin \mathrm{Sol}_1(\lambda_0, \mu_0) + B$, then ($\mathrm{iv_h}$) yields some neighborhood B_Z of 0 in Z and some $\hat{x}_0 \in S(x_0, y_0, \lambda_0)$, $x_0^* \in A(x_0, y_0, \lambda_0)$ such that

$$F(\hat{x}_0, y_0, x_0^*, \mu_0) + B_Z \subseteq -\mathrm{int}C, \tag{9}$$

or some $\hat{y}_0 \in T(x_0, y_0, \lambda_0)$, $y_0^* \in B(x_0, y_0, \lambda_0)$ such that

$$G(\hat{y}_0, x_0, y_0^*, \mu_0) + B_Z \subseteq -\mathrm{int}C. \tag{10}$$

Assume that (9) is satisfied. Taking the lower semicontinuity of S and A at (x_0, y_0, λ_0) into account one has $\hat{x}_\alpha \in S(x_\alpha, y_\alpha, \lambda_\alpha)$, $x_\alpha^* \in A(x_\alpha, y_\alpha, \lambda_\alpha)$ such that $(\hat{x}_\alpha, x_\alpha^*) \to (\hat{x}_0, x_0^*)$. Since F is $-C$-Husc at $(\hat{x}_0, y_0, x_0^*, \mu_0)$, there is some $\bar{\alpha}$ such that $F(\hat{x}_{\bar{\alpha}}, y_{\bar{\alpha}}, x_{\bar{\alpha}^*}, \mu_{\bar{\alpha}}) \subseteq -\mathrm{int}C$, which is impossible, since $(x_{\bar{\alpha}}, y_{\bar{\alpha}}) \in \mathrm{Sol}_1(\lambda_{\bar{\alpha}}, \mu_{\bar{\alpha}})$. The case of (10) is analogous. Thus $(x_0, y_0) \in \mathrm{Sol}_1(\lambda_0, \mu_0) + B \subseteq -\mathrm{int}U$. This in turn contradicts the fact that $(x_\alpha, y_\alpha) \notin -\mathrm{int}U$ for all α. □

Taking into account Proposition 3.1 in Anh and Khanh (2004) and Propositions 2.2 and 2.3 we obtain the following immediate consequence of Theorem 4.3.

Corollary 4.3 *Assume (ii_l), (iii_{hu}), and (iv_h) as in Theorem 4.3 and replace* ($\mathrm{i_{hu}}$) *by*

(i'_{hu}) *E is Husc and $E(\lambda_0)$ is compact.*

Then $Sol_1(., .)$ is Husc at (λ_0, μ_0).

The newly imposed assumption ($\mathrm{iv_h}$) cannot be dropped even for the case of quasiequilibrium problems as shown by Example 3.2 in Anh and Khanh (2004). Furthermore, for the special case of quasiequilibrium problems, Corollary 4.3 improves Theorem 3.2 in Anh and Khanh (2004) and it coincides with Theorem 3.2 in Anh and Khanh (2007a).

5 Comparison of the two solution sets

We have seen a symmetry between the sufficient conditions for the two solution sets Sol_1 and Sol_2 to be U-lower-level closed or U-upper-level closed. The following examples show that these are far from necessary conditions and the two sets may be or not be U-level closed to very different extends.

Example 5.1 (Sol_1 is continuous, Sol_2 is not lower or upper-level closed). Let $X = Y = Z = R, \Lambda \equiv M = [0, 1], K = D = R, C = R_+, A(x, y, \lambda) = \{x\}, B(x, y, \lambda) = \{y\}, S(x, y, \lambda) = T(x, y, \lambda) = [-1, 1], \lambda_0 = 0$ and

$$F(x, y, x^*, \lambda) = \begin{cases} (1 + x^*)[-1, 1] & \text{if} \quad \lambda = 0, \\ (1 - x^*)[-1, 1] & \text{otherwise,} \end{cases}$$

$$G(y, x, y^*, \lambda) = \begin{cases} (1 + y^*)[-1, 1] & \text{if} \quad \lambda = 0, \\ (1 - y^*)[-1, 1] & \text{otherwise.} \end{cases}$$

It is easy to see that $Sol_1(\lambda) = [-1, 1] \times [-1, 1], \forall\lambda \in \Lambda$ and $Sol_2(0) = \{-1\} \times \{-1\}, Sol_2(\lambda) = \{1\} \times \{1\}, \forall\lambda \in (0, 1]$. So $Sol_1(.)$ is satisfied all kinds of U-semicontinuity at 0. Taking $U = R_+ \times R_+$, we see that $Sol_2(.)$ is neither U-lower-level closed at 0 nor U-Hausdorff-upper-level closed at 0. Indeed, $\forall\lambda_\alpha \to 0, Sol_2(\lambda_\alpha) = \{1\} \times \{1\} \in \mathrm{cl}U$, but $Sol_2(0) = \{-1\} \times \{-1\} \notin \mathrm{cl}U$, and with $B = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2}), Sol_2(\lambda_\alpha) \notin -\mathrm{int}U$, but $Sol_2(0) + B = (-\frac{3}{2}, -\frac{1}{2}) \times (-\frac{3}{2}, -\frac{1}{2}) \subseteq -\mathrm{int}U$.

Example 5.2 (Sol_1 is not lower-level closed, Sol_2 is continuous). Let $X, Y, Z, \Lambda, M, K, D, C, A, B, S, T, U$, and λ_0 be as in Example 5.1 and

$$F(x, y, x^*, \lambda) = \begin{cases} \{x^* - x, 1\} & \text{if} \quad \lambda = 0, \\ \{x^* - x\} & \text{otherwise,} \end{cases}$$

$$G(y, x, y^*, \lambda) = \begin{cases} \{y^* - y, 1\} & \text{if} \quad \lambda = 0, \\ \{y^* - y\} & \text{otherwise.} \end{cases}$$

One sees that $Sol_1(0) = [-1, 1] \times [-1, 1], Sol_1(\lambda) = \{1\} \times \{1\}$ for $\lambda \in (0, 1]$ and $Sol_2(\lambda) = \{1\} \times \{1\}$ for all $\lambda \in [0, 1]$. Hence $Sol_1(.)$ is not U-lower-level closed at 0 and $S_2(.)$ satisfies all kinds of U-level closedness at 0.

Example 5.3 (Sol_1 is not Hausdorff-upper-level closed, Sol_2 is continuous). Let $X, Y, Z, \Lambda, M, K, D, C, A, B, S, T, U$, and λ_0 be as in Example 5.1 and

$$F(x, y, x^*, \lambda) = \begin{cases} \{x - x^*\} & \text{if} \quad \lambda = 0, \\ \{x - x^*, 1\} & \text{otherwise,} \end{cases}$$

$$G(y, x, y^*, \lambda) = \begin{cases} \{y - y^*\} & \text{if} \quad \lambda = 0, \\ \{y - y^*, 1\} & \text{otherwise.} \end{cases}$$

Then $Sol_1(0)=\{-1\} \times \{-1\}, Sol_1(\lambda)=[-1, 1] \times [-1, 1], \forall\lambda \in (0, 1], Sol_2(\lambda)=\{-1\} \times \{-1\}, \forall\lambda \in [0, 1]$. So $Sol_2(.)$ fulfils all kinds of U-level closedness at 0. But $Sol_2(.)$ is not U-Hausdorff-upper-level closed at 0. Indeed, taking $\lambda_\alpha \to 0, B=(-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$, $Sol_1(\lambda_\alpha)=[-1, 1] \times [-1, 1] \not\subseteq -\mathrm{int}U$, and $Sol_1(0) + B=(-\frac{3}{2}, -\frac{1}{2}) \times (-\frac{3}{2}, -\frac{1}{2}) \subseteq -\mathrm{int}U$.

Being very different in general but under some connectedness assumptions the two solution sets coincide as follows.

Theorem 5.1 *Assume that* $\forall(\bar{x}, \bar{y}) \in Sol_1(\lambda_0, \mu_0), \forall x \in S(\bar{x}, \bar{y}, \lambda_0), \forall x^* \in A(\bar{x}, \bar{y}, \lambda_0), \forall y \in T(\bar{x}, \bar{y}, \lambda_0), \forall y^* \in B(\bar{x}, \bar{y}, \lambda_0), F(x, \bar{y}, x^*, \mu_0)$*, and* $G(y, \bar{x}, y^*, \mu_0)$ *are arcwisely connected and does not meet the boundary of* $-C$*. Then* $Sol_2(\lambda_0, \mu_0) = Sol_1(\lambda_0, \mu_0)$.

Proof We always have $\mathrm{Sol}_2(\lambda_0, \mu_0) \subseteq \mathrm{Sol}_1(\lambda_0, \mu_0)$. To see the reverse inclusion let $(\bar{x}, \bar{y}) \notin \mathrm{Sol}_2(\lambda_0, \mu_0)$ then $\exists x \in S(\bar{x}, \bar{y}, \lambda_0)$, $\exists x^* \in A(\bar{x}, \bar{y}, \lambda_0)$ such that,

$$\exists z_1 \in F(x, \bar{y}, x^* \mu_0),\ z_1 \in -\mathrm{int}C, \tag{11}$$

or $\exists y \in T(\bar{x}, \bar{y}, \lambda_0)$, $\exists y^* \in B(\bar{x}, \bar{y}, \lambda_0)$ such that,

$$\exists z_1' \in G(y, \bar{x}, y^* \mu_0),\ z_1' \in -\mathrm{int}C. \tag{12}$$

Suppose that $(\bar{x}, \bar{y}) \in \mathrm{Sol}_1(\lambda_0, \mu_0)$. Then, since $F(x, \bar{y}, x^*, \mu_0)$ does not meet the boundary of $-C$, $\exists z_2 \in F(x, \bar{y}, x^*, \mu_0)\backslash(-C)$. Since $F(x, \bar{y}, x^*, \mu_0)$ is arcwisely connected, there exists a continuous mapping $\varphi : [0, 1] \to F(x, \bar{y}, x^*, \mu_0)$ such that $\varphi(0) = z_1$ and $\varphi(1) = z_2$. Let $T = \{t \in (0, 1] : \varphi([t, 1]) \subseteq Z\backslash(-C)\}$ and $t_0 = \inf T$. Since $z_1 \in -\mathrm{int}C$ there is an open set A such that $A \cap F(x, \bar{y}, x^*, \mu_0)$ is arcwisely connected and $z_1 \in A \subseteq -\mathrm{int}C$. Then $\varphi^{-1}(A \cap F(x, \bar{y}, x^*, \mu_0)) \cap T = \emptyset$. Since $\varphi^{-1}(A \cap F(x, \bar{y}, x^*, \mu_0))$ is open in $[0, 1]$, it is of the form $[0, t_1)$. So it contains 0 and $0 < t_1 \le t_0$. Similarly, $t_0 < 1$. Then, for all large n, there is $t_n \in \left(t_0 - \frac{1}{n}, t_0\right]$ such that $\varphi(t_n) \in -C$. Then $\varphi(t_0) \in -C$ since $t_n \to t_0$ and $-C$ is closed. On the other hand, for all large n, there is $t_n \in \left(t_0, t_0 + \frac{1}{n}\right)$ such that $\varphi(t_n) \in Z\backslash(-C)$. So $\varphi(t_0) \in \mathrm{cl}(Z\backslash(-C))$. Thus $\varphi(t_0)$ is in the boundary of $-C$, contradicting the fact that $\varphi(t_0) \in F(x, \bar{y}, x^*, \mu_0)$. If (12) holds, we also have the same contradiction. Hence $\mathrm{Sol}_1(\lambda_0, \mu_0) = \mathrm{Sol}_2(\lambda_0, \mu_0)$. □

The examples below ensure us the essentialness of the assumptions of Theorem 5.1.

Example 5.4 Let $X, Y, Z, \Lambda, M, K, D, C, A$, and B as in Example 5.1 and $S(x, y, \lambda) = T(x, y, \lambda) = [0, 1]$, $F(x, \bar{y}, x^*, \lambda) = \{-x^*, x^*\}$, $G(y, \bar{x}, y^*, \mu) = \{1\}$. It is clear that $\mathrm{Sol}_1(\lambda) = [0, 1]\times[0, 1]$, $\forall\lambda \in \Lambda$ and $\mathrm{Sol}_2(\lambda) = \{0\}\times[0, 1]$, $\forall\lambda \in \Lambda$. Hence $\mathrm{Sol}_1(\lambda_0, \mu_0) \neq \mathrm{Sol}_2(\lambda_0, \mu_0)$, the reason is that for $(\bar{x}, \bar{y}) \in \mathrm{Sol}_1(\lambda)$, $\bar{x} \neq 0$, $F(x, \bar{y}, x^*, \lambda)$ is not arcwisely connected for some $x \in S(\bar{x}, \bar{y}, \lambda)$, $x^* \in A(\bar{x}, \bar{y}, \lambda)$.

Example 5.5 Let $X, Y, Z, \Lambda, M, K, D, C, S, T, A, B$, and G as in Example 5.4 and $F(x, \bar{y}, x^*, \lambda) = [-x^*, x^*]$. Then $\mathrm{Sol}_1(\lambda) = [0, 1]\times[0, 1]$, $\forall\lambda \in \Lambda$ and $\mathrm{Sol}_2(\lambda) = \{0\}\times[0, 1]$, $\forall\lambda \in \Lambda$. Hence $\mathrm{Sol}_1(\lambda_0, \mu_0) \neq \mathrm{Sol}_2(\lambda_0, \mu_0)$, the reason is that for $(\bar{x}, \bar{y}) \in \mathrm{Sol}_1(\lambda)$, $F(x, \bar{y}, x^*, \lambda)$ meets the boundary of $-C$.

6 Applications

Since symmetric quasiequilibrium problems contain many problems as special cases, including quasiequilibrium problems, quasivariational inequalities, quasioptimization problems, fixed point and coincidence point problems, complementarity problems, Nash equilibria problems, etc, we can derive from theorems and corollaries in Sects. 3 and 4 consequences for these special cases. In this section we discuss only some corollaries for a lower and upper bounded quasiequilibrium problem as an example. This problem, for $(\lambda, \mu) \in \Lambda \times M$, consists of

$$\text{(BQEP)} \quad \text{finding } \bar{x} \in S_1(\bar{x}, \lambda) \text{ such that } \forall y \in S_1(\bar{x}, \lambda),$$
$$\alpha \le f(\bar{x}, y, \mu) \le \beta,$$

where $S_1 : K \times \Lambda \to 2^X$, $f : K \times K \times M \to R$, $\alpha, \beta \in R : \alpha < \beta$.

Setting $X = Y, Z = R, K = D, C = R_+, S(x, y, \lambda) = T(x, y, \lambda) = S_1(x, \lambda), A(x, y, \lambda) = \{x\}, B(x, y, \lambda) = \{y\}$ and

$$F(x, \bar{y}, x^*, \mu) = f(x^*, x, \mu) - \alpha, \tag{13}$$

$$G(y, \bar{x}, y^*, \mu) = \beta - f(x^*, y, \mu), \tag{14}$$

problem (BQEP) becomes a case of problem ($SQEP_1$) (or, the same, ($SQEP_2$)).

Set $E_1 = \{x \in K \mid x \in S(x, \lambda)\}$ and $\mathrm{Sol}(\lambda, \mu)$ is solution set of (BQEP) at $(\lambda, \mu) \in \Lambda \times M$.

Let us now analyze the assumptions of the results in Sects. 3 and 4, applied to (BQEP).

For F and G given in (13) and (14) the condition that F and G are $(0, +\infty)$-lsc at (x_0, y_0, μ_0) become (in terms of f)

$$[(x_\gamma, y_\gamma, \mu_\gamma) \to (x_0, y_0, \mu_), \alpha < f(x_0, y_0, \mu_0) < \beta]$$
$$\Longrightarrow [\exists \bar{\gamma}, \alpha < f(x_{\bar{\gamma}}, y_{\bar{\gamma}}, \mu_{\bar{\gamma}}) < \beta].$$

This property is naturally called the (α, β)-boundedness of f at (x_0, y_0, μ_0).

It is clear that F and G are R_--usc or R_--Husc become that f is $(-\infty, \alpha) \cup (\beta, +\infty)$-bounded.

Similarly R_+-inclusion properties in (iii_1) and (iii_2) will be the following condition in terms of f:

$$[(x_\gamma, y_\gamma, \mu_\gamma) \to (x_0, y_0, \mu_0), \alpha \le f(x_0, y_0, \mu_0) \le \beta]$$
$$\Longrightarrow [\exists \bar{\gamma}, \alpha \le f(x_{\bar{\gamma}}, y_{\bar{\gamma}}, \mu_{\bar{\gamma}}) \le \beta],$$

which is called the $[\alpha, \beta]$-boundedness of f at (x_0, y_0, μ_0).

Note that if $f : X \to R$ is continuous at $\bar{x}$ and $\alpha, \beta \in R$ then f is both (α, β)-bounded and $(-\infty, \alpha) \cup (\beta, +\infty)$-bounded at $\bar{x}$ but f may be not $[\alpha, \beta]$-bounded at $\bar{x}$ as shown by the following example.

Example 6.1 Let $X = Y = R, f(x) = x, \alpha = 0, \beta = 1, x_0 = 0$. It is clear that f is continuous at 0 but f is not $[0, 1]$-bounded at 0. Indeed, let $x_n = -\frac{1}{n}$, one has $f(0) \in [0, 1]$ but $f(x_n) \notin [0, 1], \forall n$.

Now Theorems 3.1–3.2 and Corollaries 3.1–3.2 derive the following four corollaries, respectively.

Corollary 6.1 *For problem $(BQEP)$ assume that, for $\emptyset \neq U \subseteq X$,*

(i_l) $E_1(.)\backslash \mathrm{cl}U$ *is lsc at* λ_0;
(ii_u) S *is usc and compact valued in* $K \times \{\lambda_0\}$;
(iii_l^l) f *is* (α, β)*-bounded in* $K \times K \times \{\mu_0\}$;
(iv_1) *for each* $x \in \mathrm{Sol}(\lambda_0, \mu_0), \forall y \in S(x, \lambda), \alpha < f(x, y, \mu_0) < \beta$.

Then $Sol(., .)$ is U-lower-level closed at (λ_0, μ_0).

Corollary 6.2 *Assume (ii_u)–(iv_1) of Corollary 6.1. Assume further that*

(i_l') E *is lsc at* λ_0.

Then $Sol(., .)$ is lsc at (λ_0, μ_0).

Corollary 6.3 *Assume (i_l) and (ii_u) as in Corollary 6.1 and replace $(iii_l{}^l)$ and (iv_1) by* *(iii$_1$)* f *is* $[\alpha, \beta]$*-bounded in* $K \times K \times \{\mu_0\}$.

Then $Sol(.,.)$ *is* U*-lower-level closed at* (λ_0, μ_0).

Corollary 6.4 *Assume* (ii_u) *and* (iii_1) *as in Corollary 6.3 and replace* (i_l) *by*

(i'_l) E *is lsc at* λ_0.

Then $Sol(.,.)$ *is lsc at* (λ_0, μ_0).

The next four corollaries are direct consequences of Theorems 3.5–3.6 and Corollaries 3.5–3.6, respectively.

Corollary 6.5 *Assume* (ii_u), (iii^l_l), *and* (iv_1) *of Corollary 6.1. Assume further, for* $\emptyset \neq U \subseteq X$, *that*

(i) E *is lsc with respect to* $\mathrm{int}U$ *at* λ_0 *and* $E(\lambda_0)\backslash\mathrm{int}U$ *is compact*;
(ii) $S(.,.,\lambda_0)$ *is lsc*;
(iii) $f(.,.,\lambda_0)$ *is* $(-\infty, \alpha) \cup (\beta, +\infty)$*-bounded in* $K \times K$.

Then $Sol(.,.)$ *is* U*-Hausdorff-lower-level closed at* (λ_0, μ_0).

Corollary 6.6 *Assume all assumptions as in Corollary 6.5 but* (i), *and replace* (i) *by*

(i') E *is lsc at* λ_0 *and* $E(\lambda_0)$ *is compact.*

Then $\mathrm{Sol}(.,.)$ *is Hlsc at* $(\lambda_0.\mu_0)$.

Corollary 6.7 *Assume all assumptions of Corollary 6.5 but* (iii^l_l) *and* (iv_1). *Assume further that*

(iii_1) f *is* $[\alpha, \beta]$*-bounded in* $K \times K \times \{\mu_0\}$.

Then $Sol(.,.)$ *is* U*-Hausdorff-lower-level closed at* (λ_0, μ_0).

Corollary 6.8 *Assume all assumptions of Corollary 6.7 but* (i), *and replace* (i) *by*

(i') E *is lsc at* λ_0 *and* $E(\lambda_0)$ *is compact.*

Then $Sol(.,.)$ *is Hlsc at* $(\lambda_0.\mu_0)$.

It is easy to see that for the solution set of problem (BQEP) the upper semicontinuity and Hausdorff upper semicontinuity coincide. The following two corollaries are direct consequences of the results in Sect. 4.

Corollary 6.9 *Assume that, for* $\emptyset \neq U \subseteq X$,

(i_u) $E(.)\backslash -\mathrm{int}U$ *is usc and* $E(\lambda_0)\backslash -\mathrm{int}U$ *is compact*;
(ii$_1$) S *is lsc in* $K \times \{\lambda_0\}$;
(iii^u_u) f *is* $(-\infty, \alpha) \cup (\beta, +\infty)$*-bounded in* $K \times K \times \{\mu_0\}$.

Then $Sol(.,.)$ *is* U*-upper-level closed at* (λ_0, μ_0).

Corollary 6.10 *Assume* (ii_l) *and* (iii^u_u) *as in Corollary 6.9 and replace* $(\mathrm{i_u})$ *by*

(i'_u) E *is usc and* $E(\lambda_0)$ *is compact.*

Then $Sol(.,.)$ *is both usc and closed at* (λ_0, μ_0).

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